

Onofri-type inequalities for singular Liouville equations

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Abstract

We study the blow-up behaviour of minimizing sequences for the singular Moser-Trudinger functional on compact surfaces. Assuming non-existence of minimum points, we give an estimate for the infimum value of the functional. This result can be applied to give sharp Onofri-type inequalities on the sphere in the presence of at most two singularities.

1 Introduction

Let (Σ, g) be a smooth, compact Riemannian surface; the standard Moser-Trudinger inequality (see [22], [16]) states that

$$\log \left(\frac{1}{|\Sigma|} \int_{\Sigma} e^{u-\bar{u}} dv_g \right) \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla_g u|^2 dv_g + C(\Sigma, g) \quad \forall u \in H^1(\Sigma) \quad (1)$$

where $C(\Sigma, g)$ is a constant depending only on Σ and g , and the coefficient $\frac{1}{16\pi}$ is optimal. A sharp version of (1) was proved by Onofri in [23] for the sphere endowed with the standard Euclidian metric g_0 . He identified the sharp value of C and the family of functions realizing equality, proving

$$\log \left(\frac{1}{4\pi} \int_{S^2} e^{u-\bar{u}} dv_{g_0} \right) \leq \frac{1}{16\pi} \int_{S^2} |\nabla_{g_0} u|^2 dv_{g_0} \quad (2)$$

with equality holding if and only if the metric $e^u g$ has constant positive Gaussian curvature, or, equivalently, $u = \log |\det d\varphi| + c$ with $c \in \mathbb{R}$ and φ conformal diffeomorphism of S^2 .

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The author is supported by the FIRB project *Analysis and Beyond*, by the PRINs *Variational Methods and Nonlinear PDE's* and *Variational and perturbative aspects of nonlinear differential problems* and by the Mathematics Department at the University of Warwick.

Onofri's inequality played an important role (see [12], [13]) in the variational approach to the equation

$$\Delta_{g_0} u + K e^u = 1$$

which is connected to the classical problem of prescribing the Gaussian curvature of S^2 . In this paper we will consider extensions of Onofri's result in connection with the study of the more general equation

$$-\Delta_g v = \rho \left(\frac{K e^v}{\int_{\Sigma} K e^v dv_g} - \frac{1}{|\Sigma|} \right) - 4\pi \sum_{i=1}^m \alpha_i \left(\delta_{p_i} - \frac{1}{|\Sigma|} \right), \quad (3)$$

where $K \in C^\infty(\Sigma)$ is a positive function, $\rho > 0$, $p_1, \dots, p_m \in \Sigma$ and $\alpha_1, \dots, \alpha_m \in (-1, +\infty)$. This is known as the singular Liouville equation and arises in several problems in Riemannian geometry and mathematical physics. When $(\Sigma, g) = (S^2, g_0)$ and $\rho = 8\pi + 4\pi \sum_{i=1}^m \alpha_i$, solutions of (3) provide metrics on S^2 with prescribed Gaussian curvature K and conical singularities of angle $2\pi(1 + \alpha_i)$ (or of order α_i) in p_i , $i = 1, \dots, m$ (see for example [3], [14], [27]). Equation (3) also appears in the description of Abelian Chern-Simons vortices in superconductivity and Electroweak theory ([17], [25]). We refer to [4], [9], [10], [11], [21], for some recent existence results. Systems of Liouville equations have applications in the description of holomorphic curves in \mathbb{CP}^n ([6], [8]) and in the nonabelian Chern-Simons theory which might have applications in high temperature superconductivity (see [26] and references therein). Denoting by G_p the Green's function at p , namely the solution of

$$\begin{cases} -\Delta_g G_p = \delta_p - \frac{1}{|\Sigma|} \\ \int_{\Sigma} G_p dv_g = 0 \end{cases},$$

the change of variables

$$u = v + 4\pi \sum_{i=1}^m \alpha_i G_{p_i}$$

transforms (3) into

$$-\Delta_g u = \rho \left(\frac{h e^u}{\int_{\Sigma} h e^u dv_g} - \frac{1}{|\Sigma|} \right) \quad (4)$$

where

$$h = K \prod_{1 \leq i \leq m} e^{-4\pi \alpha_i G_{p_i}} \quad (5)$$

satisfies

$$h(p) \approx d(p, p_i)^{2\alpha_i} \text{ for } p \approx p_i. \quad (6)$$

In [27], studying curvature functions for surfaces with conical singularities, Troyanov proved that if $h \in C^\infty(\Sigma \setminus \{p_1, \dots, p_m\})$ is a positive function satisfying (6), then

$$\log \left(\frac{1}{|\Sigma|} \int_{\Sigma} h e^{u-\bar{u}} dv_g \right) \leq \frac{1}{16\pi \min \left\{ 1, 1 + \min_{1 \leq i \leq m} \alpha_i \right\}} \int_{\Sigma} |\nabla_g u|^2 dv_g + C(\Sigma, g, h). \quad (7)$$

The optimal constant $C(\Sigma, g, h)$ can be obtained by minimizing the functional

$$J_{\bar{\rho}}(u) = \frac{1}{2} \int_{\Sigma} |\nabla_g u|^2 dv_g + \frac{\bar{\rho}}{|\Sigma|} \int_{\Sigma} u dv_g - \bar{\rho} \log \left(\frac{1}{|\Sigma|} \int_{\Sigma} h e^u dv_g \right),$$

where $\bar{\rho} = \min \left\{ 1, 1 + \min_{1 \leq i \leq m} \alpha_i \right\}$. In this paper we will assume non-existence of minimum points for $J_{\bar{\rho}}$ and exploit known blow-up results ([1], [2], [5]) to describe the behaviour of a suitable minimizing sequence and compute $\inf_{H^1(\Sigma)} J_{\bar{\rho}}$. The same technique was used by Ding, Jost, Li and Wang [15] to give an existence result for (3) in the regular case. From their proof it follows that if $\alpha_i = 0 \forall i$ and there is no minimum point for $J_{\bar{\rho}}$, then

$$\inf_{H^1(\Sigma)} J_{\bar{\rho}} = -8\pi \left(1 + \log \left(\frac{\pi}{|\Sigma|} \right) + \max_{p \in \Sigma} \{4\pi A(p) + \log h(p)\} \right)$$

where $A(p)$ is the value in p of the regular part of G_p . Here we extend this result to the general case proving:

Theorem 1.1. *Assume that h satisfies (5) with $K \in C^\infty(\Sigma)$, $K > 0$, $\alpha_i \in (-1, +\infty) \setminus \{0\}$, and that there is no minimum point of $J_{\bar{\rho}}$. If $\alpha := \min_{1 \leq i \leq m} \alpha_i < 0$, then*

$$\inf_{H^1(\Sigma)} J_{\bar{\rho}} = -8\pi(1 + \alpha) \left(1 + \log \left(\frac{\pi}{|\Sigma|} \right) + \max_{1 \leq i \leq m, \alpha_i = \alpha} \left\{ 4\pi A(p_i) + \log \left(\frac{K(p_i)}{1 + \alpha} \prod_{j \neq i} e^{-4\pi \alpha_j G_{p_j}(p_i)} \right) \right\} \right)$$

while if $\alpha > 0$

$$\inf_{H^1(\Sigma)} J_{\bar{\rho}} = -8\pi \left(1 + \log \left(\frac{\pi}{|\Sigma|} \right) + \max_{p \in \Sigma \setminus \{p_1, \dots, p_m\}} \{4\pi A(p) + \log h(p)\} \right).$$

In the last part of the paper we consider the case of the standard sphere with $K \equiv 1$ and at most two singularities. When $m = 1$ a simple Kazdan-Warner type identity proves non-existence of solutions for (4). Thus, one can apply theorem 1.1 to obtain the following sharp version of (7):

Theorem 1.2. *If $h = e^{-4\pi\alpha_1 G_{p_1}}$ with $\alpha_1 \neq 0$, then $\forall u \in H^1(S^2)$*

$$\log\left(\frac{1}{4\pi} \int_{S^2} h e^{u-\bar{u}} dv_{g_0}\right) < \frac{1}{16\pi \min\{1, 1 + \alpha_1\}} \int_{S^2} |\nabla u|^2 dv_{g_0} + \max\{\alpha_1, -\log(1 + \alpha_1)\}.$$

The same non-existence argument works for $m = 2$, $\min\{\alpha_1, \alpha_2\} < 0$ and $\alpha_1 \neq \alpha_2$ if the singularities are located in two antipodal points.

Theorem 1.3. *Assume $h = e^{-4\pi\alpha_1 G_{p_1} - 4\pi\alpha_2 G_{p_2}}$ with $p_2 = -p_1$, $\alpha_1 = \min\{\alpha_1, \alpha_2\} < 0$ and $\alpha_1 \neq \alpha_2$; then $\forall u \in H^1(S^2)$*

$$\log\left(\frac{1}{4\pi} \int_{S^2} h e^{u-\bar{u}} dv_{g_0}\right) < \frac{1}{16\pi(1 + \alpha_1)} \int_{S^2} |\nabla u|^2 dv_{g_0} + \alpha_2 - \log(1 + \alpha_1).$$

When $\alpha_1 = \alpha_2 < 0$ theorem 1.1 cannot be directly applied because (4) has solutions. However, it is possible to use a stereographic projection and a classification result in [24] to find an explicit expression for the solutions. In particular a direct computation allows to prove that all the solutions are minimum points of $J_{\bar{\rho}}$ and to find the value of $\min_{H^1(S^2)} J_{\bar{\rho}}$.

Theorem 1.4. *Assume $h = e^{-4\pi\alpha(G_{p_1} + G_{p_2})}$ with $\alpha < 0$ and $p_1 = -p_2$; then $\forall u \in H^1(S^2)$ we have*

$$\log\left(\frac{1}{4\pi} \int_{S^2} h e^{u-\bar{u}} dv_{g_0}\right) \leq \frac{1}{16\pi(1 + \alpha)} \int_{S^2} |\nabla u|^2 dv_{g_0} + \alpha - \log(1 + \alpha).$$

Moreover the following conditions are equivalent:

- u realises equality.
- If π denotes the stereographic projection from p_1 then

$$u \circ \pi^{-1}(y) = 2 \log\left(\frac{(1 + |y|^2)^{1+\alpha}}{1 + e^\lambda |y|^{2(1+\alpha)}}\right) + c$$

for some $\lambda, c \in \mathbb{R}$.

- $h e^u g_0$ is a metric with constant positive Gaussian curvature and conical singularities of order α_i in p_i , $i = 1, 2$.

This is a generalization of Onofri's inequality (2) for metrics with two conical singularities.

2 Preliminaries and blow up analysis

Let (Σ, g) be a smooth compact, connected, Riemannian surface and let $S := \{p_1, \dots, p_m\}$ be a finite subset of Σ . Let us consider a function h satisfying (5) with $K \in C^\infty(\Sigma)$, $K > 0$ and $\alpha_i \in (-1, +\infty) \setminus \{0\}$. In order to distinguish the singular points of h from the regular ones, we introduce a singularity index function

$$\beta(p) := \begin{cases} \alpha_i & \text{if } p = p_i \\ 0 & \text{if } p \notin S \end{cases}.$$

We will denote $\alpha := \min_{p \in \Sigma} \beta(p) = \min \left\{ \min_{1 \leq i \leq m} \alpha_i, 0 \right\}$ the minimum singularity order. We shall consider the functional

$$J_\rho(u) = \frac{1}{2} \int_{\Sigma} |\nabla_g u|^2 dv_g + \frac{\rho}{|\Sigma|} \int_{\Sigma} u dv_g - \rho \log \left(\frac{1}{|\Sigma|} \int_{\Sigma} h e^u dv_g \right). \quad (8)$$

Our goal is to give a sharp version of (7) finding the explicit value of

$$C(\Sigma, g, h) = -\frac{1}{8\pi(1+\alpha)} \inf_{u \in H^1(\Sigma)} J_{8\pi(1+\alpha)}(u). \quad (9)$$

To simplify the notations we will denote $\bar{\rho} := 8\pi(1+\alpha)$, $\rho_\varepsilon = \bar{\rho} - \varepsilon$, $J_\varepsilon := J_{\rho_\varepsilon}$ and $J := J_{\bar{\rho}}$. From (7) it follows that $\forall \varepsilon > 0$ the functional J_ε is coercive and, by direct methods, it is possible to find a function $u_\varepsilon \in H^1(\Sigma)$ satisfying

$$J_\varepsilon(u_\varepsilon) = \inf_{u \in H^1(\Sigma)} J_\varepsilon(u) \quad (10)$$

and

$$-\Delta_g u_\varepsilon = \rho_\varepsilon \left(\frac{h e^{u_\varepsilon}}{\int_{\Sigma} h e^{u_\varepsilon} dv_g} - \frac{1}{|\Sigma|} \right). \quad (11)$$

Since J_ε is invariant under addition of constants $\forall \varepsilon > 0$, we may also assume

$$\int_{\Sigma} h e^{u_\varepsilon} dv_g = 1. \quad (12)$$

Remark 2.1. $u_\varepsilon \in C^{0,\gamma}(\Sigma) \cap W^{1,s}(\Sigma)$ for some $\gamma \in (0, 1)$ and $s > 2$.

Proof. It is easy to see that $h \in L^q(\Sigma)$ for some $q > 1$ ($q = +\infty$ if $\alpha = 0$ and $q < -\frac{1}{\alpha}$ for $\alpha < 0$). Applying locally Remarks 2 and 5 in [7] one can show that $u_\varepsilon \in L^\infty(\Sigma)$ so $-\Delta u_\varepsilon \in L^q(\Sigma)$ and by standard elliptic estimates $u_\varepsilon \in W^{2,q}(\Sigma)$. Since $q > 1$ the conclusion follows by Sobolev's embedding theorems. \square

The behaviour of u_ε is described by the following concentration-compactness result:

Proposition 2.1. *Let u_n be a sequence of solutions of*

$$-\Delta_g u_n = V_n e^{u_n} - \psi_n$$

where $\|\psi_n\|_{L^s(\Sigma)} \leq C$ for some $s > 1$, and

$$V_n = K_n \prod_{1 \leq i \leq m} e^{-4\pi\alpha_i G_{p_i}}$$

with $K_n \in C^\infty(\Sigma)$, $0 < a \leq K_n \leq b$ and $\alpha_i > -1$, $i = 1, \dots, m$. Then there exists a subsequence u_{n_k} of u_n such that one of the following holds:

- i. u_{n_k} is uniformly bounded in $L^\infty(\Sigma)$;
- ii. $u_{n_k} \rightarrow -\infty$ uniformly on Σ ;
- iii. there exist a finite blow-up set $B = \{q_1, \dots, q_l\} \subseteq \Sigma$ and a corresponding family of sequences $\{q_k^j\}_{k \in \mathbb{N}}$, $j = 1, \dots, l$ such that $q_k^j \xrightarrow{k \rightarrow \infty} q_j$ and $u_{n_k}(q_k^j) \xrightarrow{k \rightarrow \infty} +\infty$ $j = 1, \dots, l$. Moreover $u_{n_k} \xrightarrow{k \rightarrow \infty} -\infty$ uniformly on compact subsets of $\Sigma \setminus B$ and $V_{n_k} e^{u_{n_k}} \rightarrow \sum_{j=1}^l \beta_j \delta_{q_j}$ weakly in the sense of measures where $\beta_j = 8\pi(1 + \beta(q_j))$ for $j = 1, \dots, l$.

A proof of proposition 2.1 in the regular case can be found in [19] while the general case is a consequence of the results in [1] and [5]. In our analysis we will also need the following local version of proposition 2.1 proved by Li and Shafrir ([20]):

Proposition 2.2. *Let Ω be an open domain in \mathbb{R}^2 and v_n be a sequence satisfying $\|e^{v_n}\|_{L^1(\Omega)} \leq C$ and*

$$-\Delta v_n = V_n e^{v_n}$$

where $0 \leq V_n \in C_0(\overline{\Omega})$ and $V_n \rightarrow V$ uniformly in $\overline{\Omega}$. If v_n is not uniformly bounded from above on compact subset of Ω , then $V_n e^{v_n} \rightarrow 8\pi \sum_{i=1}^l m_j \delta_{q_j}$ as measures, with $q_j \in \Omega$ and $m_j \in \mathbb{N}^+$, $j = 1, \dots, l$.

Applying proposition 2.1 to u_ε under the additional condition (12) we obtain that either u_ε is uniformly bounded in $L^\infty(\Sigma)$ or its blows-up set contains a single point p such that $\beta(p) = \alpha$. In the first case, one can use elliptic estimates to find uniform bounds on u_ε in $W^{2,q}(\Sigma)$, for some $q > 1$; consequently, a subsequence of u_ε converges in $H^1(\Sigma)$ to a function $u \in H^1(\Sigma)$ that is a minimum point of J and a solution of (4) for $\rho = \bar{\rho}$. We now focus on the second case, that is

$$\lambda_\varepsilon := \max_{\Sigma} u_\varepsilon = u_\varepsilon(p_\varepsilon) \rightarrow +\infty \quad \text{and} \quad p_\varepsilon \rightarrow p \quad \text{with} \quad \beta(p) = \alpha. \quad (13)$$

By proposition 2.1 we also get:

Lemma 2.1. *If u_ε satisfies (11), (12) and (13), then, up to subsequences,*

1. $\rho_\varepsilon h e^{u_\varepsilon} \rightharpoonup \bar{\rho} \delta_p$;
2. $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} -\infty$ uniformly in Ω , $\forall \Omega \subset\subset \Sigma \setminus \{p\}$;
3. $\bar{u}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} -\infty$;
4. *There exist $\gamma \in (0, 1)$, $s > 2$ such that $u_\varepsilon - \bar{u}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \bar{\rho} G_p$ in $C^{0,\gamma}(\bar{\Omega}) \cap W^{1,s}(\Omega) \forall \Omega \subset\subset \Sigma \setminus \{p\}$;*
5. ∇u_ε is bounded in $L^q(\Sigma) \forall q \in (1, 2)$.

Proof. 1., 2. and 3. are direct consequences of proposition 2.1. To prove 4. we consider Green's representation formula

$$u_\varepsilon(x) - \bar{u}_\varepsilon = \rho_\varepsilon \int_{\Sigma} G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y).$$

We stress that Green's function has the following properties:

- $|G_x(y)| \leq C_1(1 + |\log d(x, y)|) \forall x, y \in \Sigma, x \neq y$.
- $|\nabla_g^x G_x(y)| \leq \frac{C_2}{d(x, y)} \forall x, y \in \Sigma, x \neq y$.
- $G_x(y) = G_y(x) \forall x, y \in \Sigma, x \neq y$.

Take $q > 1$ such that $h \in L^q(\Sigma)$. The first property also yields

$$\sup_{x \in \Sigma} \|G_x\|_{L^{q'}(\Sigma)} \leq C_3. \quad (14)$$

Let us fix $\delta > 0$ such that $B_{3\delta}(p) \subset \Sigma \setminus \Omega$ and take a cut-off function φ such that $\varphi \equiv 1$ in $B_\delta(p)$ and $\varphi \equiv 0$ in $\Sigma \setminus B_{2\delta}(p)$.

$$u_\varepsilon(x) - \bar{u}_\varepsilon = \rho_\varepsilon \int_{\Sigma} \varphi(y) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) + \rho_\varepsilon \int_{\Sigma} (1 - \varphi(y)) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y).$$

By (14) and 2. we have

$$\begin{aligned} \left| \int_{\Sigma} (1 - \varphi(y)) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) \right| &\leq \int_{\Sigma \setminus B_\delta(p)} |G_x(y)| |h(y) e^{u_\varepsilon(y)}| dv_g(y) \leq \\ &\leq C_3 \|h\|_{L^q(\Sigma)} \|e^{u_\varepsilon}\|_{L^\infty(\Sigma \setminus B_\delta(p))} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

By 1. and the smoothness of φG_x for $x \in \overline{\Omega}$ and $y \in \Sigma$ we get

$$\int_{\Sigma} \varphi(y) G_x(y) h(y) e^{u_{\varepsilon}(y)} dv_g(y) \xrightarrow{\varepsilon \rightarrow 0} \varphi(p) G_x(p) = G_p(x)$$

uniformly for $x \in \Omega$. Similarly we have

$$\nabla_g u_{\varepsilon}(x) = \rho_{\varepsilon} \int_{\Sigma} \varphi(y) \nabla_g^x G_x(y) h(y) e^{u_{\varepsilon}(y)} dv_g(y) + \rho_{\varepsilon} \int_{\Sigma} (1 - \varphi(y)) \nabla_g^x G_x(y) h(y) e^{u_{\varepsilon}(y)} dv_g(y)$$

with

$$\int_{\Sigma} \varphi(y) \nabla_g^x G_x(y) h(y) e^{u_{\varepsilon}(y)} dv_g(y) \xrightarrow{k \rightarrow \infty} \nabla_g^x G_p(x)$$

uniformly in Ω and, assuming $q \in (1, 2)$, by the Hardy-Littlewood-Sobolev inequality

$$\begin{aligned} & \int_{\Sigma} \left(\int_{\Sigma} (1 - \varphi(y)) \nabla_g^x G_x(y) h(y) e^{u_{\varepsilon}(y)} dv_g(y) \right)^s dv_g(x) \leq \\ & \leq C_2^s \int_{\Sigma} \left(\int_{\Sigma \setminus B_{\delta}(p)} \frac{h(y) e^{u_{\varepsilon}(y)}}{d(x, y)} dv_g(y) \right)^s dv_g(x) \leq C \|h\|_{L^q(\Sigma)}^s \|e^{u_{\varepsilon}}\|_{L^{\infty}(\Sigma \setminus B_{\delta}(p))}^s \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

where

$$\frac{1}{s} = \frac{1}{q} - \frac{1}{2}.$$

Note that $q > 1$ implies $s > 2$. Finally, to prove 5., we shall observe that for any $1 < q < 2$ there exists a positive constant C_q such that

$$\int_{\Sigma} \varphi dv_g = 0 \quad \text{and} \quad \int_{\Sigma} |\nabla_g \varphi|^{q'} dv_g \leq 1 \quad \implies \quad \|\varphi\|_{\infty} \leq C_q.$$

Hence $\forall \varphi \in W^{1, q'}(\Sigma)$

$$\int_{\Sigma} \nabla_g u_{\varepsilon} \cdot \nabla_g \varphi dv_g = - \int_{\Sigma} \Delta u_{\varepsilon} \varphi dv_g \leq C_q \|\Delta u_{\varepsilon}\|_{L^1(\Sigma)} \leq \tilde{C}_q$$

so that

$$\|\nabla u_{\varepsilon}\|_{L^q} \leq \sup \left\{ \int_{\Sigma} \nabla_g u_{\varepsilon} \cdot \nabla_g \varphi dv_g : \varphi \in W^{1, q'}(\Sigma), \|\nabla \varphi\|_{L^{q'}} \leq 1 \right\} \leq \tilde{C}_q.$$

□

We now focus on the behaviour of u_{ε} near the blow-up point. First we consider the case $\alpha < 0$. Let us fix a system of normal coordinates in a small ball $B_{\delta}(p)$, with p corresponding to 0 and p_{ε} corresponding to x_{ε} . We define

$$\varphi_{\varepsilon}(x) := u_{\varepsilon}(t_{\varepsilon}x) - \lambda_{\varepsilon}, \quad t_{\varepsilon} := e^{-\frac{\lambda_{\varepsilon}}{2(1+\alpha)}}. \quad (15)$$

Lemma 2.2. *If $\alpha < 0$, $\frac{|x_\varepsilon|}{t_\varepsilon}$ is bounded.*

Proof. We define

$$\psi_\varepsilon(x) = u_\varepsilon(|x_\varepsilon|x) + 2(1 + \alpha) \log |x_\varepsilon| + s_\varepsilon(|x_\varepsilon|x)$$

where $s_\varepsilon(x)$ is the solution of

$$\begin{cases} -\Delta s_\varepsilon = \frac{\rho_\varepsilon}{|\Sigma|} & \text{in } B_\delta(0) \\ s_\varepsilon = 0 & \text{if } |x| = \delta \end{cases}.$$

The function ψ_ε satisfies

$$-\Delta \psi_\varepsilon = |x_\varepsilon|^{-2\alpha} \rho_\varepsilon h(|x_\varepsilon|x) e^{-s_\varepsilon(|x_\varepsilon|x)} e^{\psi_\varepsilon} = V_\varepsilon e^{\psi_\varepsilon}$$

in $B_{\frac{\delta}{|x_\varepsilon|}}(0)$. We stress that, by standard elliptic estimates, s_ε is uniformly bounded in $C^1(\overline{B_\delta})$ and that G_p has the expansion

$$G_p(x) = -\frac{1}{2\pi} \log |x| + A(p) + O(|x|) \quad (16)$$

in $B_\delta(0)$. Thus

$$\begin{aligned} |x_\varepsilon|^{-2\alpha} h(|x_\varepsilon|x) e^{-s_\varepsilon(|x_\varepsilon|x)} &= |x_\varepsilon|^{-2\alpha} e^{2\alpha \log(|x_\varepsilon|x) - 4\pi\alpha A(p) + O(|x_\varepsilon||x|)} e^{-s_\varepsilon(|x_\varepsilon|x)} K(|x_\varepsilon|x) \prod_{1 \leq i \leq m, p_i \neq p} e^{-4\pi\alpha_i G_{p_i}(|x_\varepsilon|x)} = \\ &= |x|^{2\alpha} e^{-4\pi\alpha A(p)} e^{O(|x_\varepsilon||x|)} e^{-s_\varepsilon(|x_\varepsilon|x)} K(|x_\varepsilon|x) \prod_{1 \leq i \leq m, p_i \neq p} e^{-4\pi\alpha_i G_{p_i}(|x_\varepsilon|x)} = |x|^{2\alpha} \tilde{h}(|x_\varepsilon|x) \end{aligned}$$

where $\tilde{h} \in C^1(\overline{B_\delta})$. In particular V_ε is uniformly bounded in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$. If there existed a subsequence such that $\frac{|x_\varepsilon|}{t_\varepsilon} \rightarrow +\infty$ then

$$\psi_\varepsilon\left(\frac{x_\varepsilon}{|x_\varepsilon|}\right) = 2(1 + \alpha) \log\left(\frac{|x_\varepsilon|}{t_\varepsilon}\right) + s_\varepsilon(x_\varepsilon) \rightarrow +\infty,$$

so $y_0 := \lim_{\varepsilon \rightarrow 0} \frac{x_\varepsilon}{|x_\varepsilon|}$ would be a blow up point for ψ_ε . Since $y_0 \neq 0$, applying proposition 2.2 to ψ_ε in a small ball $B_r(y_0)$ we would get

$$\liminf_{\varepsilon \rightarrow 0} \int_{B_r(y_0)} V_\varepsilon e^{\psi_\varepsilon} dx \geq 8\pi.$$

But this would be in contradiction to (12) since

$$\int_{B_r(y_0)} V_\varepsilon e^{\psi_\varepsilon} dx = \int_{B_r(y_0)} \rho_\varepsilon |x_\varepsilon|^{-2\alpha} h(|x_\varepsilon|x) e^{-s_\varepsilon(|x_\varepsilon|x)} e^{\psi_\varepsilon} dx \leq \rho_\varepsilon \int_{B_\delta(p)} h e^{u_\varepsilon} dv_g \leq 8\pi(1 + \alpha) < 8\pi.$$

□

Lemma 2.3. Assume $\alpha < 0$. Then, possibly passing to a subsequence, φ_ε converges uniformly on compact subsets of \mathbb{R}^2 and in $H_{loc}^1(\mathbb{R}^2)$ to

$$\varphi_0(x) := -2 \log \left(1 + \frac{\pi c(p)}{1 + \alpha} |x|^{2(1+\alpha)} \right)$$

where $c(p) = K(p)e^{-4\pi\alpha A(p)} \prod_{1 \leq i \leq m, p_i \neq p} e^{-4\pi\alpha_i G_{p_i}(p)}$.

Proof. The function φ_ε is defined in $B_\varepsilon = B_{\frac{\delta}{t_\varepsilon}}(0)$ and satisfies

$$-\Delta \varphi_\varepsilon = t_\varepsilon^2 \rho_\varepsilon \left(h(t_\varepsilon x) e^{\varphi_\varepsilon} e^{\lambda_\varepsilon} - \frac{1}{|\Sigma|} \right) = t_\varepsilon^{-2\alpha} \rho_\varepsilon h(t_\varepsilon x) e^{\varphi_\varepsilon} - \frac{t_\varepsilon^2 \rho_\varepsilon}{|\Sigma|}$$

and

$$t_\varepsilon^{-2\alpha} \int_{B_{\frac{\delta}{t_\varepsilon}}} h(t_\varepsilon x) e^{\varphi_\varepsilon} \leq 1.$$

As in the previous proof we have

$$\begin{aligned} t_\varepsilon^{-2\alpha} h(t_\varepsilon x) &= t_\varepsilon^{-2\alpha} e^{2\alpha \log(t_\varepsilon |x|) - 4\pi\alpha A(p) + O(t_\varepsilon |x|)} K(t_\varepsilon x) \prod_{1 \leq i \leq m, p_i \neq p} e^{-4\pi\alpha_i G_{p_i}(t_\varepsilon x)} = \\ &= |x|^{2\alpha} e^{-4\pi\alpha A(p)} e^{O(t_\varepsilon |x|)} K(t_\varepsilon x) \prod_{1 \leq i \leq m, p_i \neq p} e^{-4\pi\alpha_i G_{p_i}(t_\varepsilon x)} \xrightarrow{\varepsilon \rightarrow 0} c(p) |x|^{2\alpha} \end{aligned}$$

in $L_{loc}^q(\mathbb{R}^2)$ for some $q > 1$. Fix $R > 0$ and let ψ_ε be the solution of

$$\begin{cases} -\Delta \psi_\varepsilon = t_\varepsilon^{-2\alpha} \rho_\varepsilon h(t_\varepsilon x) e^{\varphi_\varepsilon} - \frac{t_\varepsilon^2 \rho_\varepsilon}{|\Sigma|} & \text{in } B_R(0) \\ \psi_\varepsilon = 0 & \text{su } \partial B_R(0) \end{cases}.$$

Since $\Delta \psi_\varepsilon$ is bounded in $L^q(B_R(0))$ with $q > 1$, elliptic regularity shows that ψ_ε is bounded in $W^{2,q}(B_R(0))$ and by Sobolev's embeddings we may extract a subsequence such that ψ_ε converges in $H^1(B_R(0)) \cap C^{0,\lambda}(B_R(0))$. The function $\xi_\varepsilon = \varphi_\varepsilon - \psi_\varepsilon$ is harmonic in B_R and bounded from above. Furthermore $\xi_\varepsilon \left(\frac{x_\varepsilon}{t_\varepsilon} \right) = -\psi_\varepsilon \left(\frac{x_\varepsilon}{t_\varepsilon} \right)$ is bounded from below, hence by Harnack inequality ξ_ε is uniformly bounded in $C^2(\overline{B_{\frac{R}{2}}}(0))$. Thus φ_ε is bounded in $W^{2,q}(B_{\frac{R}{2}})$ and we can extract a subsequence converging in $H^1(B_{\frac{R}{2}}) \cap C^{0,\lambda}(B_{\frac{R}{2}})$. Using a diagonal argument we find a subsequence for which φ_ε converges in $H_{loc}^1(\mathbb{R}^2) \cap C_{loc}^{0,\lambda}(\mathbb{R}^2)$ to a function φ_0 solving

$$-\Delta \varphi_0 = 8\pi(1 + \alpha)c(p)|x|^{2\alpha} e^{\varphi_0}$$

on \mathbb{R}^2 with

$$\int_{\mathbb{R}^2} |x|^{2\alpha} e^{\varphi_0(x)} dx < \infty.$$

The classification result in [24] yields

$$\varphi_0(x) = -2 \log \left(1 + \frac{\pi e^\lambda c(p)}{1 + \alpha} |x|^{2(1+\alpha)} \right) + \lambda$$

for some $\lambda \in \mathbb{R}$. To conclude the proof it remains to note that, since 0 is the unique maximum point of φ_0 , the uniform convergence of φ_ε implies $\frac{x_\varepsilon}{t_\varepsilon} \rightarrow 0$ and $\lambda = 0$. \square

As in [15], to give a lower bound on $J_\varepsilon(u_\varepsilon)$ we need the following estimate from below for u_ε :

Lemma 2.4. *Fix $R > 0$ and define $r_\varepsilon = t_\varepsilon R$. If $\alpha < 0$ and u_ε satisfies (11), (12), (13), then*

$$u_\varepsilon \geq \bar{\rho} G_p - \lambda_\varepsilon - \bar{\rho} A(p) + 2 \log \left(\frac{R^{2(1+\alpha)}}{1 + \frac{\pi c(p)}{1+\alpha} R^{2(1+\alpha)}} \right) + o_\varepsilon(1)$$

in $\Sigma \setminus B_{r_\varepsilon}(p)$.

Proof. $\forall C > 0$ we have

$$-\Delta_g(u_\varepsilon - \bar{\rho} G_p - C) = \rho_\varepsilon \left(h e^{u_\varepsilon} - \frac{1}{|\Sigma|} \right) + \frac{\bar{\rho}}{|\Sigma|} = \rho_\varepsilon h e^{u_\varepsilon} + \frac{\varepsilon}{|\Sigma|} \geq 0.$$

Let us consider normal coordinates near p . We know that

$$G_p(x) = -\frac{1}{2\pi} \log |x| + A(p) + O(|x|),$$

so by lemma 2.3 if $x = t_\varepsilon y$ with $|y| = R$ we have

$$\begin{aligned} u_\varepsilon(x) - \bar{\rho} G_p &= \varphi_\varepsilon(y) + \lambda_\varepsilon + 4(1 + \alpha) \log(t_\varepsilon R) - \bar{\rho} A(p) + o_\varepsilon(1) = \\ &= -2 \log \left(1 + \frac{\pi c(p)}{1 + \alpha} R^{2(1+\alpha)} \right) - \lambda_\varepsilon + \log R^{4(1+\alpha)} - \bar{\rho} A(p) + o_\varepsilon(1). \end{aligned}$$

Thus, taking

$$C_\varepsilon = -\lambda_\varepsilon - \bar{\rho} A(p) + 2 \log \left(\frac{R^{2(1+\alpha)}}{1 + \frac{\pi c(p)}{1+\alpha} R^{2(1+\alpha)}} \right) + o_\varepsilon(1)$$

we have $u_\varepsilon - \bar{\rho} G_p - C_\varepsilon \geq 0$ on $\partial B_{r_\varepsilon}(p)$ and the conclusion follows from the maximum principle. \square

As a consequence we also have

Lemma 2.5. $t_\varepsilon^2 \bar{u}_\varepsilon \rightarrow 0$.

Proof. By lemma 2.3

$$\int_{B_{t_\varepsilon}(p)} u_\varepsilon dv_g = t_\varepsilon^2 \int_{B_1(0)} \varphi_\varepsilon(y) dy + \lambda_\varepsilon |B_{t_\varepsilon}| = o_\varepsilon(1).$$

and by the previous lemma

$$\lambda_\varepsilon |\Sigma| \geq \int_{\Sigma \setminus B_{t_\varepsilon}(p)} u_\varepsilon \geq \bar{\rho} \int_{\Sigma \setminus B_{t_\varepsilon}(p)} G_p dv_g - \lambda_\varepsilon |\Sigma \setminus B_{t_\varepsilon}(p)| + O(1).$$

Thus $\frac{|\bar{u}_\varepsilon|}{\lambda_\varepsilon}$ is bounded and, since $\lambda_\varepsilon t_\varepsilon^2 = o_\varepsilon(1)$, we get the conclusion. \square

The case $\alpha = 0$ can be studied in a similar way. The main difference is that, since we do not know whether $\frac{|x_\varepsilon|}{t_\varepsilon}$ is bounded, we have to center the scaling in p_ε and not in p . Note that $\beta(p) = 0$ means that $p \in \Sigma \setminus S$ is a regular point of h .

Lemma 2.6. Assume that $\alpha = 0$ and that u_ε satisfies (11), (12) and (13). In normal coordinates near p define

$$\psi_\varepsilon(x) = u_\varepsilon(x_\varepsilon + t_\varepsilon x) - \lambda_\varepsilon \quad \text{where} \quad t_\varepsilon = e^{-\frac{\lambda_\varepsilon}{2}}.$$

Then

1. ψ_ε converges in $C_{loc}^1(\mathbb{R}^2)$ to

$$\psi_0(x) = -2 \log(1 + \pi h(p) |x|^2)$$

2. $\forall R > 0$ one has

$$u_\varepsilon \geq 8\pi G_{p_\varepsilon} - \lambda_\varepsilon - 8\pi A(p) + 2 \log \left(\frac{R^2}{1 + \pi h(p) R^2} \right) + o_\varepsilon(1)$$

in $\Sigma \setminus B_{Rt_\varepsilon}(p_\varepsilon)$;

3. $t_\varepsilon^2 \bar{u}_\varepsilon \rightarrow 0$.

3 A lower bound

In this section and in the next one we present the proof of theorem 1.1. We begin by giving an estimate from below of $\inf_{H^1(\Sigma)} J$. As before we consider u_ε satisfying (10), (11), (12), and (13). Again we will focus on the case $\alpha < 0$ since the computation for $\alpha = 0$ is equivalent to the one in [15]. We consider normal coordinates in a small ball $B_\delta(p)$ and assume that G_p

has the expansion (16) in $B_\delta(p)$. Let t_ε be defined as in (15), then $\forall R > 0$ we shall consider the decomposition

$$\int_{\Sigma} |\nabla_g u_\varepsilon|^2 dv_g = \int_{\Sigma \setminus B_\delta(p)} |\nabla_g u_\varepsilon|^2 dv_g + \int_{B_\delta \setminus B_{r_\varepsilon}(p)} |\nabla_g u_\varepsilon|^2 dv_g + \int_{B_{r_\varepsilon}(p)} |\nabla_g u_\varepsilon|^2 dv_g.$$

On $\Sigma \setminus B_\delta(p)$ we can use lemma 2.1 and an integration by parts to obtain:

$$\begin{aligned} \int_{\Sigma \setminus B_\delta} |\nabla_g u_\varepsilon|^2 dv_g &= \bar{\rho}^2 \int_{\Sigma \setminus B_\delta} |\nabla_g G_p|^2 dv_g + o_\varepsilon(1) = \\ &= -\frac{\bar{\rho}^2}{|\Sigma|} \int_{\Sigma \setminus B_\delta} G_p dv_g - \bar{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1) = \\ &= -\bar{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1) + o_\delta(1). \end{aligned} \quad (17)$$

On $B_{r_\varepsilon}(p)$ the convergence result for the scaling (15) stated in lemma 2.3 yields

$$\int_{B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g = \int_{B_R(0)} |\nabla \varphi_0|^2 dx + o_\varepsilon(1) = 2\bar{\rho} \left(\log \left(1 + \frac{\pi c(p)}{1+\alpha} R^{2(1+\alpha)} \right) - 1 \right) + o_\varepsilon(1) + o_R(1). \quad (18)$$

For the remaining term we can use (11) and lemma 2.1 to obtain

$$\begin{aligned} \int_{B_\delta \setminus B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g &= \rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} u_\varepsilon dv_g - \frac{\rho_\varepsilon}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g + \int_{\partial B_\delta} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\sigma_g = \\ &= \rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} u_\varepsilon dv_g - \frac{\rho_\varepsilon}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g + \bar{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \\ &\quad + \bar{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1). \end{aligned} \quad (19)$$

By lemma 2.4 and (12) we get

$$\begin{aligned} \rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} u_\varepsilon dv_g &\geq \rho_\varepsilon \bar{\rho} \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} G_p dv_g - \rho_\varepsilon \lambda_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g + O_R(1) \rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g = \\ &= \rho_\varepsilon \bar{\rho} \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} G_p dv_g - \rho_\varepsilon \lambda_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g + o_\varepsilon(1). \end{aligned} \quad (20)$$

Again by (11) and lemma 2.1

$$\begin{aligned}
\rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} G_p dv_g &= \int_{B_\delta \setminus B_{r_\varepsilon}} G_p \left(-\Delta u_\varepsilon + \frac{\rho_\varepsilon}{|\Sigma|} \right) dv_g = \\
&= -\frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g + \int_{\partial B_\delta} u_\varepsilon \frac{\partial G_p}{\partial n} - G_p \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \int_{\partial B_{r_\varepsilon}} G_p \frac{\partial u_\varepsilon}{\partial n} - u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g + o_\delta(1) = \\
&= -\frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g + \bar{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial G_p}{\partial n} d\sigma_g + \int_{\partial B_{r_\varepsilon}} G_p \frac{\partial u_\varepsilon}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g + \\
&\quad + o_\varepsilon(1) + o_\delta(1),
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\rho_\varepsilon \lambda_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g &= -\lambda_\varepsilon \int_{\partial B_\delta \setminus B_{r_\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} (Vol(B_\delta) - Vol(B_{r_\varepsilon})) = \\
&= -\lambda_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \lambda_\varepsilon \int_{\partial B_{r_\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} Vol(B_\delta) + o_\varepsilon(1).
\end{aligned} \tag{22}$$

Using (19), (20), (21) and (22) we get

$$\begin{aligned}
\int_{B_\delta \setminus B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g &\geq -(16\pi(1+\alpha) - \varepsilon) \frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g - \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} Vol(B_\delta) + \\
&\quad + \bar{\rho} \bar{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial G_p}{\partial n} d\sigma_g + \lambda_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \bar{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \\
&\quad + \bar{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g - \bar{\rho} \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} (u_\varepsilon - \bar{\rho} G_p + \lambda_\varepsilon) \frac{\partial u_\varepsilon}{\partial n} + \\
&\quad + o_\varepsilon(1) + o_\delta(1).
\end{aligned} \tag{23}$$

By lemmas 2.1 and 2.5 we can say that

$$\int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g = \int_{B_\delta \setminus B_{r_\varepsilon}} (u_\varepsilon - \bar{u}_\varepsilon) dv_g + \bar{u}_\varepsilon (Vol(B_\delta) - Vol(B_{r_\varepsilon})) = \bar{u}_\varepsilon Vol(B_\delta) + o_\delta(1) + o_\varepsilon(1).$$

Using Green's formula

$$\bar{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial G_p}{\partial n} d\sigma_g = -\bar{u}_\varepsilon \int_{\Sigma \setminus B_\delta} \Delta_g G_p dv_g = -\bar{u}_\varepsilon \left(1 - \frac{Vol(B_\delta)}{|\Sigma|} \right).$$

Similarly

$$\int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g = - \int_{\Sigma \setminus B_\delta} \Delta u_\varepsilon dv_g = \int_{\Sigma \setminus B_\delta} \rho_\varepsilon \left(h e^{u_\varepsilon} - \frac{1}{|\Sigma|} \right) dv_g \geq -\rho_\varepsilon \left(1 - \frac{Vol(B_\delta)}{|\Sigma|} \right)$$

and

$$\begin{aligned}
\bar{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g &= \bar{u}_\varepsilon \rho_\varepsilon e^{\bar{u}_\varepsilon} \int_{\Sigma \setminus B_\delta(p)} h e^{u_\varepsilon - \bar{u}_\varepsilon} dv_g - \bar{u}_\varepsilon \rho_\varepsilon \left(1 - \frac{\text{Vol}(B_\delta)}{|\Sigma|}\right) = \\
&= -\bar{u}_\varepsilon \rho_\varepsilon \left(1 - \frac{\text{Vol}(B_\delta)}{|\Sigma|}\right) + o_\varepsilon(1).
\end{aligned}$$

Lemma 2.3 yields

$$\begin{aligned}
\int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g &= \lambda_\varepsilon \int_{\partial B_\varepsilon} \frac{\partial G_p}{\partial n} d\sigma_g + t_\varepsilon \int_{\partial B_R(0)} \varphi_\varepsilon \frac{\partial G_p}{\partial n} (t_\varepsilon x) (1 + o_\varepsilon(1)) d\sigma = \\
&= -\lambda_\varepsilon \left(1 - \frac{\text{Vol}(B_{r_\varepsilon})}{|\Sigma|}\right) + t_\varepsilon \int_{\partial B_R(0)} \varphi_0 \left(-\frac{1}{2\pi t_\varepsilon R} + O(1)\right) d\sigma = \\
&= -\lambda_\varepsilon + 2 \log \left(1 + \frac{\pi c(p)}{1 + \alpha} R^{2(1+\alpha)}\right) + o_\varepsilon(1)
\end{aligned}$$

and the estimate in lemma 2.4 gives

$$\begin{aligned}
& - \int_{\partial B_{r_\varepsilon}} (u_\varepsilon - \bar{\rho} G_p + \lambda_\varepsilon) \frac{\partial u_\varepsilon}{\partial n} d\sigma_g \geq \\
& \geq \left(2 \log \left(\frac{R^{2(1+\alpha)}}{1 + \frac{\pi c(p)}{(1+\alpha)} R^{2(1+\alpha)}}\right) - \bar{\rho} A(p)\right) \frac{8\pi^2 c(p) R^{2(1+\alpha)}}{\left(1 + \frac{\pi c(p)}{1+\alpha} R^{2(1+\alpha)}\right)} + o_\varepsilon(1) = \\
& = -\bar{\rho}^2 A(p) - 2 \bar{\rho} \log \left(\frac{\pi c(p)}{1 + \alpha}\right) + o_\varepsilon(1) + o_R(1).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{B_\delta \setminus B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g &\geq -(16\pi(1+\alpha) - \varepsilon) \bar{u}_\varepsilon + \varepsilon \lambda_\varepsilon + \bar{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g + \\
& - 2\bar{\rho} \log \left(1 + \frac{\pi c(p)}{1 + \alpha} R^{2(1+\alpha)}\right) - \bar{\rho}^2 A(p) - 2\bar{\rho} \log \left(\frac{\pi c(p)}{1 + \alpha}\right) + \\
& + o_\varepsilon(1) + o_\delta(1) + o_R(1).
\end{aligned} \tag{24}$$

By (17), (18) and (24) we can therefore conclude

$$\int_\Sigma |\nabla_g u_\varepsilon|^2 dv_g \geq -(16\pi(1+\alpha) - \varepsilon) \bar{u}_\varepsilon + \varepsilon \lambda_\varepsilon - \bar{\rho}^2 A(p) - 2\bar{\rho} \log \left(\frac{\pi c(p)}{1 + \alpha}\right) - 2\bar{\rho} +$$

$$+ o_\varepsilon(1) + o_\delta(1) + o_R(1),$$

so that

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &\geq \frac{\varepsilon}{2}(\lambda_\varepsilon - \bar{u}_\varepsilon) - \frac{\bar{\rho}^2}{2}A(p) - \bar{\rho} \log \left(\frac{\pi c(p)}{1+\alpha} \right) - \bar{\rho} + \rho_\varepsilon \log |\Sigma| + o_\varepsilon(1) + o_\delta(1) + o_R(1) \geq \\ &\geq -\bar{\rho} \left(4\pi(1+\alpha)A(p) + 1 + \log \left(\frac{\pi c(p)}{1+\alpha} \right) - \log |\Sigma| \right) + o_\varepsilon(1) + o_\delta(1) + o_R(1). \end{aligned}$$

As $\varepsilon, \delta \rightarrow 0$ and $R \rightarrow \infty$ we obtain

$$\begin{aligned} \inf_{H^1(\Sigma)} J &\geq -\bar{\rho} \left(4\pi(1+\alpha)A(p) + 1 + \log \left(\frac{\pi c(p)}{1+\alpha} \right) - \log |\Sigma| \right) = \\ &= -\bar{\rho} \left(1 + \log \frac{\pi}{|\Sigma|} + 4\pi A(p) + \log \left(\frac{K(p)}{1+\alpha} \prod_{q \in S, q \neq p} e^{-4\pi\beta(q)G_q(p)} \right) \right). \end{aligned} \tag{25}$$

Using lemma 2.6 it is possible to prove that (25) holds even for $\alpha = 0$. About the blow-up point p we only know that $\beta(p) = \alpha$, so we have proved

Proposition 3.1. *If J has no minimum point, then*

$$\inf_{H^1(\Sigma)} J \geq -\bar{\rho} \left(1 + \log \frac{\pi}{|\Sigma|} + \max_{p \in \Sigma, \beta(p)=\alpha} \left\{ 4\pi A(p) + \log \left(\frac{K(p)}{1+\alpha} \prod_{q \in S, q \neq p} e^{-4\pi\beta(q)G_q(p)} \right) \right\} \right).$$

Notice that, if $\alpha < 0$, the set

$$\{p \in \Sigma : \beta(p) = \alpha\} = \{p_i : i \in \{1, \dots, m\}, \alpha_i = \alpha\}$$

is finite, while if $\alpha = 0$

$$\{p \in \Sigma : \beta(p) = \alpha\} = \Sigma \setminus S.$$

Although this set is not finite, the maximum in the above expression is still well defined since the function

$$p \mapsto 4\pi A(p) + \log \left(K(p) \prod_{q \in S} e^{-4\pi\beta(q)G_q(p)} \right) = 4\pi A(p) + \log h(p)$$

is continuous on $\Sigma \setminus S$ and approaches $-\infty$ near S .

4 An estimate from above

In order to complete the proof of theorem 1.1 we need to exhibit a sequence $\varphi_\varepsilon \in H^1(\Sigma)$ such that

$$J(\varphi_\varepsilon) \longrightarrow -\bar{\rho} \left(1 + \log \frac{\pi}{|\Sigma|} + \max_{p \in \Sigma, \beta(p)=\alpha} \left\{ 4\pi A(p) + \log \left(\frac{K(p)}{1+\alpha} \prod_{q \in S, q \neq p} e^{-4\pi\beta(q)G_q(p)} \right) \right\} \right)$$

Let us define $r_\varepsilon := \gamma_\varepsilon \varepsilon^{\frac{1}{2(1+\alpha)}}$ where γ_ε is chosen so that

$$\gamma_\varepsilon \rightarrow +\infty, \quad r_\varepsilon^2 \log \varepsilon \longrightarrow 0, \quad r_\varepsilon^2 \log(1 + \gamma_\varepsilon^{2(1+\alpha)}) \longrightarrow 0. \quad (26)$$

Let $p \in \Sigma$ be such that $\beta(p) = \alpha$ and

$$4\pi A(p) + \log \left(\frac{K(p)}{1+\alpha} \prod_{q \in S, q \neq p} e^{-4\pi\beta(q)G_q(p)} \right) = \max_{\xi \in \Sigma, \beta(\xi)=\alpha} \left\{ 4\pi A(\xi) + \log \left(\frac{K(\xi)}{1+\alpha} \prod_{q \in S, q \neq \xi} e^{-4\pi\beta(q)G_q(\xi)} \right) \right\}$$

and consider a cut-off function η_ε such that $\eta_\varepsilon \equiv 1$ in $B_{r_\varepsilon}(p)$, $\eta_\varepsilon \equiv 0$ in $\Sigma \setminus B_{2r_\varepsilon}(p)$ and $|\nabla_g \eta_\varepsilon| = O(r_\varepsilon^{-1})$. Define

$$\varphi_\varepsilon(x) = \begin{cases} -2 \log(\varepsilon + r_\varepsilon^{2(1+\alpha)}) + \log \varepsilon & r \leq r_\varepsilon \\ \bar{\rho}(G_p - \eta_\varepsilon \sigma) + C_\varepsilon + \log \varepsilon & r \geq r_\varepsilon \end{cases}$$

where $r = d(x, p)$, $\sigma(x) = O(r)$ is defined by

$$G_p(x) = -\frac{1}{2\pi} \log r + A(p) + \sigma(x), \quad (27)$$

and

$$C_\varepsilon = -2 \log \left(\frac{1 + \gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}} \right) - \bar{\rho} A(p).$$

In the case $\alpha_i = 0 \forall i$, a similar family of functions was used in [15] to give an existence result for (4) by proving, under some strict assumptions on h , that

$$\inf_{H^1(\Sigma)} J_{\bar{\rho}} < -8\pi \left(1 + \log \left(\frac{\pi}{|\Sigma|} \right) + \max_{p \in \Sigma} \{ 4\pi A(p) + \log h(p) \} \right).$$

Here we only prove large inequality but we have no extra assumptions on h . Taking normal coordinates in a neighbourhood of p it is simple to verify that

$$\int_{B_{r_\varepsilon}} |\nabla_g \varphi_\varepsilon|^2 dv_g = 16\pi(1+\alpha) \left(\log \left(1 + \gamma_\varepsilon^{2(1+\alpha)} \right) + \frac{1}{1 + \gamma_\varepsilon^{2(1+\alpha)}} - 1 \right) + o_\varepsilon(1) =$$

$$= 16\pi(1+\alpha)\left(\log\left(1+\gamma_\varepsilon^{2(1+\alpha)}\right)-1\right)+o_\varepsilon(1).$$

By our definition of φ_ε

$$\int_{\Sigma \setminus B_{r_\varepsilon}} |\nabla_g \varphi_\varepsilon|^2 dv_g = \bar{\rho}^2 \left(\int_{\Sigma \setminus B_{r_\varepsilon}} |\nabla_g G_p|^2 dv_g + \int_{\Sigma \setminus B_{r_\varepsilon}} |\nabla_g(\eta_\varepsilon \sigma)|^2 dv_g - 2 \int_{\Sigma \setminus B_{r_\varepsilon}} \nabla_g G_p \cdot \nabla_g(\eta_\varepsilon \sigma) dv_g \right)$$

and by the properties of η_ε

$$\int_{\Sigma \setminus B_{r_\varepsilon}} |\nabla_g(\eta_\varepsilon \sigma)|^2 dv_g = \int_{B_{2r_\varepsilon} \setminus B_{r_\varepsilon}} |\nabla_g \eta_\varepsilon|^2 \sigma^2 + 2\eta_\varepsilon \sigma \nabla_g \eta_\varepsilon \cdot \nabla_g \sigma + \eta_\varepsilon^2 |\nabla_g \sigma|^2 dv_g = O(r_\varepsilon^2).$$

Hence, integrating by parts and using (27), one has

$$\begin{aligned} \int_{\Sigma \setminus B_{r_\varepsilon}} |\nabla_g \varphi_\varepsilon|^2 dv_g &= \bar{\rho}^2 \left(\int_{\Sigma \setminus B_{r_\varepsilon}} |\nabla G_p|^2 dv_g - 2 \int_{\Sigma \setminus B_{r_\varepsilon}} \nabla_g G_p \cdot \nabla_g(\eta_\varepsilon \sigma) dv_g \right) + o_\varepsilon(1) = \\ &= -\bar{\rho}^2 \left(\frac{1}{|\Sigma|} \int_{\Sigma \setminus B_{r_\varepsilon}} (G_p - 2\eta_\varepsilon \sigma) dv_g + \int_{\partial B_{r_\varepsilon}} (G_p - 2\eta_\varepsilon \sigma) \frac{\partial G_p}{\partial n} d\sigma_g \right) + o_\varepsilon(1) = \\ &= -\bar{\rho}^2 \int_{\partial B_{r_\varepsilon}} (G_p - 2\sigma) \frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1) = \\ &= -\bar{\rho}^2 \int_{\partial B_{r_\varepsilon}} \left(-\frac{1}{2\pi} \log(r_\varepsilon) + A(p) - \sigma \right) \left(-\frac{1}{2\pi r_\varepsilon} + \nabla \sigma \right) (1 + O(r_\varepsilon^2)) d\sigma + o_\varepsilon(1) = \\ &= -\bar{\rho}^2 \int_{\partial B_{r_\varepsilon}} \left(\frac{\log r_\varepsilon}{4\pi^2 r_\varepsilon} - \frac{1}{2\pi r_\varepsilon} A(p) + O(\log r_\varepsilon) + O(1) \right) d\sigma + o_\varepsilon(1) = \\ &= -\frac{\bar{\rho}^2}{2\pi} \log(\gamma_\varepsilon \varepsilon^{\frac{1}{2(1+\alpha)}}) + \bar{\rho}^2 A(p) + o_\varepsilon(1) = \\ &= -2\bar{\rho} \left(\log \gamma_\varepsilon^{2(1+\alpha)} + \log \varepsilon - 4\pi(1+\alpha)A(p) \right) + o_\varepsilon(1). \end{aligned}$$

Thus

$$\begin{aligned} \int_{\Sigma} |\nabla_g \varphi_\varepsilon|^2 dv_g &= 2\bar{\rho} \left(\log \left(\frac{1+\gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}} \right) - 1 + 4\pi(1+\alpha)A(p) - \log \varepsilon \right) + o_\varepsilon(1) = \\ &= -2\bar{\rho} (1 - 4\pi(1+\alpha)A(p) + \log \varepsilon) + o_\varepsilon(1). \end{aligned} \tag{28}$$

Similarly one has

$$\int_{B_{r_\varepsilon}} \varphi_\varepsilon dv_g = |B_{r_\varepsilon}| \log \varepsilon - 4\pi \int_0^{r_\varepsilon} r \log(\varepsilon + r^{2(1+\alpha)}) (1 + o_\varepsilon(1)) dr =$$

$$\begin{aligned}
&= |B_{r_\varepsilon}| \log \varepsilon - 2\pi r_\varepsilon^2 \log \varepsilon - 4\pi \int_0^{r_\varepsilon} r \log \left(1 + \frac{r^{2(1+\alpha)}}{\varepsilon}\right) (1 + o_\varepsilon(1)) dr = \\
&= O(r_\varepsilon^2 \log \varepsilon) - 4\pi \int_0^1 r_\varepsilon^2 s \log \left(1 + \gamma_\varepsilon^{2(1+\alpha)} s^{2(1+\alpha)}\right) (1 + o_\varepsilon(1)) dr = \\
&= O(r_\varepsilon^2 \log \varepsilon) + O(r_\varepsilon^2 \log(1 + \gamma_\varepsilon^{2(1+\alpha)})) = o_\varepsilon(1)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Sigma \setminus B_{r_\varepsilon}} \varphi_\varepsilon dv_g &= \bar{\rho} \int_{\Sigma \setminus B_{r_\varepsilon}} (G_p - \eta_\varepsilon \sigma) dv_g + (C_\varepsilon + \log \varepsilon) |\Sigma \setminus B_{r_\varepsilon}(p)| = \\
&= |\Sigma| \log \varepsilon - \bar{\rho} |\Sigma| A(p) + o_\varepsilon(1)
\end{aligned}$$

so that

$$\frac{1}{|\Sigma|} \int_{\Sigma} \varphi_\varepsilon dv_g = \log \varepsilon - \bar{\rho} A(p) + o_\varepsilon(1). \quad (29)$$

To compute the integral of the exponential term we fix a small $\delta > 0$ and observe that

$$\int_{\Sigma} h e^{\varphi_\varepsilon} dv_g = \tilde{h}(p) \int_{B_{r_\varepsilon}} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g + \int_{B_{r_\varepsilon}} (\tilde{h} - \tilde{h}(p)) e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g + \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{\varphi_\varepsilon} dv_g + \int_{\Sigma \setminus B_\delta} h e^{\varphi_\varepsilon} dv_g$$

where $\tilde{h} = h e^{4\pi\alpha G_p} = K \prod_{q \in S, q \neq p} e^{-4\pi\beta(q) G_q}$. For the first term we have

$$\begin{aligned}
\int_{B_{r_\varepsilon}} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g &= \varepsilon \int_{B_{r_\varepsilon}} e^{2\alpha \log r - 4\pi\alpha A(p) - 4\pi\alpha \sigma} e^{-2 \log(\varepsilon + r^{2(1+\alpha)})} dv_g = \\
&= \varepsilon e^{-4\pi\alpha A(p)} \int_{B_{r_\varepsilon}} \frac{r^{2\alpha}}{(\varepsilon + r^{2(1+\alpha)})^2} (1 + o_\varepsilon(1)) dv_g = \\
&= \frac{\pi e^{-4\pi\alpha A(p)}}{1 + \alpha} \frac{\gamma_\varepsilon^{2(1+\alpha)}}{1 + \gamma_\varepsilon^{2(1+\alpha)}} (1 + o_\varepsilon(1)) = \\
&= \frac{\pi e^{-4\pi\alpha A(p)}}{1 + \alpha} + o_\varepsilon(1). \quad (30)
\end{aligned}$$

Since \tilde{h} is smooth in a neighbourhood of p we obtain

$$\int_{B_{r_\varepsilon}} (\tilde{h} - \tilde{h}(p)) e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g = o_\varepsilon(1) \int_{B_{r_\varepsilon}} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g = o_\varepsilon(1) \quad (31)$$

and

$$\left| \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{\varphi_\varepsilon} dv_g \right| = \left| \int_{B_\delta \setminus B_{r_\varepsilon}} \tilde{h} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g \right| \leq \sup_{B_\delta} |\tilde{h}| \int_{B_\delta \setminus B_{r_\varepsilon}} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g =$$

$$\begin{aligned}
&= \varepsilon e^{C_\varepsilon} \sup_{B_\delta} |\tilde{h}| \int_{B_\delta \setminus B_{r_\varepsilon}} e^{4\pi(2+\alpha)G_p} e^{-\bar{\rho}\eta_\varepsilon \sigma} dv_g = \\
&= O(\varepsilon) \int_{B_\delta \setminus B_{r_\varepsilon}} e^{4\pi(2+\alpha)G_p} dx = O(\varepsilon) \int_{B_\delta \setminus B_{r_\varepsilon}} \frac{1}{|x|^{2(2+\alpha)}} dx = \\
&= O(\varepsilon) \left(\frac{1}{r_\varepsilon^{2(1+\alpha)}} - \frac{1}{\delta^{2(1+\alpha)}} \right) = O\left(\frac{1}{\gamma_\varepsilon^{2(1+\alpha)}} \right) + O(\varepsilon) = o_\varepsilon(1). \tag{32}
\end{aligned}$$

Finally

$$\int_{\Sigma \setminus B_\delta} h e^{\varphi_\varepsilon} dv_g = \varepsilon e^{C_\varepsilon} \int_{\Sigma \setminus B_\delta} h e^{\bar{\rho} G_p} dv_g = O(\varepsilon) \tag{33}$$

so by (30), (31), (32) and (33) we have

$$\int_{\Sigma} h e^{\varphi_\varepsilon} dv_g = \frac{\pi \tilde{h}(p) e^{-4\pi \alpha A(p)}}{1 + \alpha} + o_\varepsilon(1). \tag{34}$$

Using (28), (29) and (34) we get

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} J(\varphi_\varepsilon) &= -\bar{\rho} \left(1 + 4\pi A(p) + \log \left(\frac{1}{|\Sigma|} \frac{\pi \tilde{h}(p)}{1 + \alpha} \right) \right) = \\
&= -\bar{\rho} \left(1 + \log \frac{\pi}{|\Sigma|} + \max_{\xi \in \Sigma, \beta(\xi) = \alpha} \left\{ 4\pi A(\xi) + \log \left(\frac{K(\xi)}{1 + \alpha} \prod_{q \in S, q \neq \xi} e^{-4\pi \beta(q) G_q(\xi)} \right) \right\} \right).
\end{aligned}$$

This, together with proposition 3.1, completes the proof of theorem 1.1.

5 Onofri's inequalities on S^2

In this section we will consider the special case of the standard sphere (S^2, g_0) with $m \leq 2$ and $K \equiv 1$. We fix $\alpha_1, \alpha_2 \in \mathbb{R}$ with $-1 < \alpha_1 \leq \alpha_2$ and as before we consider the singular weight

$$h = e^{-4\pi \alpha_1 G_{p_1} - 4\pi \alpha_2 G_{p_2}}.$$

In order to apply theorem 1.1 and obtain sharp versions of (7), we need to study the existence of minimum points for the functional J . Let us fix a system of coordinates (x_1, x_2, x_3) on \mathbb{R}^3 such that $p_1 = (0, 0, 1)$. If $\min\{\alpha_1, \alpha_2\} \geq 0$, h is smooth in S^2 and the Kazdan-Warner identity (see [18]) states that any solution of (4) has to satisfy

$$\int_{S^2} \nabla h \cdot \nabla x_i e^u dv_{g_0} = \left(2 - \frac{\rho}{4\pi} \right) \int_{S^2} h e^u x_i dv_{g_0} \quad i = 1, 2, 3.$$

We claim that if $p_2 = -p_1$ the same identity holds, at least in the x_3 -direction, even when h is singular.

Lemma 5.1. *Let u be a solution of (4) on S^2 , then there exist $C, \delta_0 > 0$ such that*

- $|\nabla u(x)| \leq Cd(x, p_i)^{2\alpha_i+1}$ *if $\alpha_i < -\frac{1}{2}$;*
- $|\nabla u(x)| \leq C(-\log d(x, p_i))$ *if $\alpha_i = -\frac{1}{2}$;*
- $|\nabla u(x)| \leq C$ *if $\alpha_i > -\frac{1}{2}$;*

for $0 < d(x, p_i) < \delta_0$, $i = 1, 2$.

Proof. Let us fix $0 < r_0 < \frac{1}{2} \min\{\frac{\pi}{2}, d(p_1, p_2)\}$ and $i \in \{1, 2\}$. If $\alpha_i > -\frac{1}{2}$ then, by standard elliptic regularity, $u \in C^1(\overline{B_{r_0}(p_i)})$ and the conclusion holds for $\delta_0 = r_0$ and $C = \|\nabla u\|_{L^\infty(B_{r_0}(p_i))}$. Let us now assume $\alpha_i \leq -\frac{1}{2}$. We know that $h(y) \leq C_1 d(y, p_i)^{2\alpha_i}$ for $y \in B_{2r_0}(p_i)$ so, if $\delta_0 < r_0$, by Green's representation formula we have

$$|\nabla u|(x) \leq \rho e^{\|u\|_\infty} \int_{S^2} \frac{h(y)}{d(x, y)} dv_{g_0}(y) \leq \frac{\rho e^{\|u\|_\infty} \|h\|_{L^1(S^2)}}{r_0} + \rho e^{\|u\|_\infty} C_1 \int_{B_{r_0}(x)} \frac{d(y, p_i)^{2\alpha_i}}{d(x, y)} dv_{g_0}(y).$$

Let π be the stereographic projection from the point $-p_i$. It is easy to check that there exist $C_2, C_3 > 0$ such that

$$C_2 d(q, q') \leq |\pi(q) - \pi(q')| \leq C_3 d(q, q')$$

$\forall q, q' \in B_{\frac{\pi}{2}}(p_i)$. Thus we have

$$\begin{aligned} \int_{B_{r_0}(x)} \frac{d(y, p_i)^{2\alpha_i}}{d(x, y)} dv_{g_0}(y) &\leq \int_{B_{\frac{\pi}{2}}(p_i)} \frac{d(y, p_i)^{2\alpha_i}}{d(x, y)} dv_{g_0}(y) \leq C_4 \int_{\{|z| \leq 1\}} \frac{|z|^{2\alpha_i}}{|\pi(x) - z|} dz = \\ &= C_4 |\pi(x)|^{2\alpha_i+1} \int_{\{|z| \leq \frac{1}{|\pi(x)|}\}} \frac{|z|^{2\alpha_i}}{\left| \frac{\pi(x)}{|\pi(x)|} - z \right|} dz \leq C_5 d(x, p_i)^{2\alpha_i+1} \int_{\{|z| \leq \frac{1}{|\pi(x)|}\}} \frac{|z|^{2\alpha_i}}{\left| \frac{\pi(x)}{|\pi(x)|} - z \right|} dz. \end{aligned}$$

Notice that

$$\begin{aligned} \int_{\{|z| \leq \frac{1}{|\pi(x)|}\}} \frac{|z|^{2\alpha_i}}{\left| \frac{\pi(x)}{|\pi(x)|} - z \right|} dz &\leq \frac{1}{2^{2\alpha_i}} \int_{\left\{ \left| \frac{\pi(x)}{|\pi(x)|} - z \right| \leq \frac{1}{2} \right\}} \frac{1}{\left| \frac{\pi(x)}{|\pi(x)|} - z \right|} dz + 2 \int_{\{|z| \leq 2\}} |z|^{2\alpha_i} dz + 2 \int_{\{2 \leq |z| \leq \frac{1}{|\pi(x)|}\}} |z|^{2\alpha_i-1} dz \leq \\ &\leq C_6 + 2 \int_{\{2 \leq |z| \leq \frac{1}{|\pi(x)|}\}} |z|^{2\alpha_i-1} dz. \end{aligned}$$

If $\alpha_i < -\frac{1}{2}$

$$\int_{\{2 \leq |z| \leq \frac{1}{|\pi(x)|}\}} |z|^{2\alpha_i-1} dz \leq C_7,$$

while if $\alpha_i = -\frac{1}{2}$

$$\int_{\{2 \leq |z| \leq \frac{1}{|\pi(x)|}\}} |z|^{2\alpha_i-1} dz = 2\pi \log \left(\frac{1}{2|\pi(x)|} \right) \leq C_8 (-\log d(x, p_i)).$$

Thus we get the conclusion for δ_0 sufficiently small. \square

In any case there exists $s \in [0, 1)$ such that

$$|\nabla u(x)| \leq C d(x, p_i)^{-s} (-\log d(x, p_i)) \quad (35)$$

for $0 < d(x, p_i) < \delta_0$, $i = 1, 2$.

Proposition 5.1. *If $p_2 = -p_1$ then any solution of (4) satisfies*

$$\int_{S^2} \nabla h \cdot \nabla x_3 e^u dv_{g_0} = \left(2 - \frac{\rho}{4\pi}\right) \int_{S^2} h e^u x_3 dv_{g_0}.$$

Proof. Without loss of generality we may assume

$$\int_{S^2} h e^u dv_{g_0} = 1. \quad (36)$$

Let us denote $S_\delta = S^2 \setminus B_\delta(p_1) \cup B_\delta(p_2)$. Since u is smooth in S_δ , multiplying (4) by $\nabla u \cdot \nabla x_3$ and integrating on S_δ we have

$$-\int_{S_\delta} \Delta u \nabla u \cdot \nabla x_3 dv_{g_0} = \rho \int_{S_\delta} \left(h e^u - \frac{1}{4\pi} \right) \nabla u \cdot \nabla x_3 dv_{g_0} \quad (37)$$

Integrating by parts we obtain

$$-\int_{S_\delta} \Delta u \nabla u \cdot \nabla x_3 dv_{g_0} = \int_{S_\delta} \nabla u \cdot \nabla (\nabla u \cdot \nabla x_3) dv_{g_0} + \sum_{i=1}^2 \int_{\partial B_\delta(p_i)} \nabla u \cdot \nabla x_3 \frac{\partial u}{\partial n} d\sigma_{g_0}$$

and by (35)

$$\left| \int_{\partial B_\delta(p_i)} \nabla u \cdot \nabla x_3 \frac{\partial u}{\partial n} d\sigma_{g_0} \right| \leq \int_{\partial B_\delta(p_i)} |\nabla u|^2 |\nabla x_3| d\sigma_{g_0} = O(\delta^{2(1-s)} \log^2 \delta) = o_\delta(1).$$

Using the identities

$$\nabla u \cdot \nabla (\nabla u \cdot \nabla x_3) = \frac{1}{2} \nabla |\nabla u|^2 \cdot \nabla x_3 - x_3 |\nabla u|^2$$

and

$$-\Delta x_3 = 2x_3,$$

and applying again (35) to estimate the boundary term, we get

$$\begin{aligned} - \int_{S_\delta} \Delta u \nabla u \cdot \nabla x_3 \, dv_{g_0} &= \int_{S_\delta} \frac{1}{2} \nabla |\nabla u|^2 \cdot \nabla x_3 \, dv_{g_0} - \int_{S_\delta} x_3 |\nabla u|^2 \, dv_{g_0} + o_\delta(1) = \\ &= -\frac{1}{2} \int_{S_\delta} \Delta x_3 |\nabla u|^2 \, dv_{g_0} - \sum_{i=1}^2 \int_{\partial B_\delta(p_i)} |\nabla u|^2 \frac{\partial x_3}{\partial n} \, d\sigma_{g_0} - \int_{S_\delta} x_3 |\nabla u|^2 \, dv_{g_0} = o_\delta(1). \end{aligned}$$

Thus (37) becomes

$$\int_{S_\delta} h e^\mu \nabla u \cdot \nabla x_3 \, dv_{g_0} - \frac{1}{4\pi} \int_{S_\delta} \nabla u \cdot \nabla x_3 \, dv_{g_0} = o_\delta(1). \quad (38)$$

Moreover

$$\begin{aligned} \int_{S_\delta} \nabla u \cdot \nabla x_3 \, dv_{g_0} &= - \int_{S_\delta} \Delta u \, x_3 \, dv_{g_0} - \sum_{i=1}^2 \int_{\partial B_\delta(p_i)} x_3 \frac{\partial u}{\partial n} \, d\sigma_{g_0} = \\ &= \rho \int_{S_\delta} \left(h e^\mu - \frac{1}{4\pi} \right) x_3 \, dv_{g_0} + O(\delta^{1-s}(-\log \delta)) = \rho \int_{S_\delta} h e^\mu x_3 \, dv_{g_0} + o_\delta(1) \end{aligned}$$

and

$$\begin{aligned} \int_{S_\delta} h e^\mu \nabla u \cdot \nabla x_3 \, dv_{g_0} &= \int_{S_\delta} \nabla e^\mu \cdot h \nabla x_3 \, dv_{g_0} = - \int_{S_\delta} e^\mu \operatorname{div}(h \nabla x_3) \, dv_{g_0} - \sum_{i=1}^2 \int_{\partial B_\delta(p_i)} h e^\mu \frac{\partial x_3}{\partial n} \, d\sigma_{g_0} = \\ &= - \int_{S_\delta} \nabla h \cdot \nabla x_3 \, e^\mu \, dv_{g_0} + 2 \int_{S_\delta} h e^\mu x_3 \, dv_{g_0} + O(\delta^{2(1+\alpha)}). \end{aligned}$$

Thus by (38) we have

$$\int_{S_\delta} \nabla h \cdot \nabla x_3 \, e^\mu \, dv_{g_0} = \left(2 - \frac{\rho}{4\pi} \right) \int_{S_\delta} h e^\mu x_3 \, dv_{g_0} + o_\delta(1).$$

Since u is continuous on S^2 and $h, \nabla h \cdot \nabla x_3 \in L^1(S^2)$ as $\delta \rightarrow 0$ we get the conclusion. \square

Remark 5.1. In this proof there is no need to assume $K \equiv 1$.

Assuming $p_1 = (0, 0, 1)$ and $p_2 = (0, 0, -1)$, one may easily verify that

$$G_{p_1}(x) = -\frac{1}{4\pi} \log(1 - x_3) - \frac{1}{4\pi} \log\left(\frac{e}{2}\right)$$

and

$$G_{p_2}(x) = -\frac{1}{4\pi} \log(1 + x_3) - \frac{1}{4\pi} \log\left(\frac{e}{2}\right),$$

so that

$$\nabla h \cdot \nabla x_3 = -4\pi h(\alpha_1 \nabla G_1 + \alpha_2 \nabla G_2) \cdot \nabla x_3 = (\alpha_2 - \alpha_1)h - (\alpha_1 + \alpha_2)hx_3.$$

Thus we can rewrite the identity in proposition 5.1 as

$$\alpha_2 - \alpha_1 = \left(2 - \frac{\rho}{4\pi} + \alpha_1 + \alpha_2\right) \int_{S^2} h e^u x_3 \, dv_{g_0}. \quad (39)$$

Proof of theorem 1.2. Assume $m = 1$ (i.e. $\alpha_2 = 0$). We claim that equation (4) has no solutions for $\rho = \bar{\rho} = 8\pi(1 + \min\{0, \alpha_1\})$, unless $\alpha_1 = 0$. Indeed if u were a solution of (4) satisfying (36), then applying (39) with $\rho = \bar{\rho}$ we would get

$$-\alpha_1 = (\alpha_1 - 2 \min\{0, \alpha_1\}) \int_{S^2} h e^u x_3 \, dv_{g_0}$$

so that, if $\alpha_1 \neq 0$,

$$\left| \int_{S^2} h e^u x_3 \, dv_{g_0} \right| = 1.$$

This contradicts (4). In particular we proved non-existence of minimum points for $J_{\bar{\rho}}$ so we can exploit theorem 1.1 and (9) to prove that (7) holds with

$$C = \max_{p \in S^2, \beta(p) = \alpha} \left\{ \log \left(\frac{1}{1 + \alpha} \prod_{q \in S, q \neq p} e^{-4\pi\beta(q)G_q(p)} \right) \right\}.$$

If $\alpha_1 < 0$ one has

$$C = -\log(1 + \alpha_1).$$

If $\alpha_1 > 0$,

$$C = \max_{p \in S^2 \setminus \{p_1\}} \{-4\pi\alpha_1 G_{p_1}(p)\} = -4\pi\alpha_1 G_{p_1}(p_2) = \alpha_1.$$

□

Proof of theorem 1.3. As in the previous proof, applying (39) with $\rho = \bar{\rho} = 8\pi(1 + \alpha_1)$, we obtain that any critical point of (4) for which (36) holds has to satisfy

$$\alpha_2 - \alpha_1 = (\alpha_2 - \alpha_1) \int_{S^2} h e^u x_3 \, dv_{g_0}.$$

Since $\alpha_1 \neq \alpha_2$ one has

$$\int_{S^2} h e^u x_3 \, dv_{g_0} = 1$$

that is impossible. Thus $J_{\bar{\rho}}$ has no critical point and by theorem 1.1 one has

$$C = \log \left(\frac{1}{1 + \alpha_1} e^{-4\pi\alpha_2 G_{p_2}(p_1)} \right) = \alpha_2 - \log(1 + \alpha_1).$$

□

Now we assume $\alpha_1 = \alpha_2 < 0$. In this case identity (39) gives no useful condition. Let us denote by π the stereographic projection from the point p_1 . It is easy to verify that u satisfies (4) and (36) if and only if

$$v := u \circ \pi^{-1} + (1 + \alpha) \log \left(\frac{4}{(1 + |y|^2)^2} \right) + 2\alpha \log \left(\frac{e}{2} \right)$$

solves

$$-\Delta_{\mathbb{R}^2} v = 8\pi(1 + \alpha)|y|^{2\alpha} e^v \quad (40)$$

in \mathbb{R}^2 and

$$\int_{\mathbb{R}^2} |y|^{2\alpha} e^v dy = 1.$$

As we pointed out in the proof of lemma 2.3, equation (40) has a one-parameter family of solutions:

$$v_\lambda(y) = -2 \log \left(1 + \frac{\pi}{1 + \alpha} e^l |y|^{2(1+\alpha)} \right)$$

$l \in \mathbb{R}$. Thus we have a corresponding family $\{u_{\lambda,c}\}$ of critical points of $J_{\bar{\rho}}$ given by the expression

$$u_{\lambda,c} \circ \pi^{-1}(y) = 2 \log \left(\frac{(1 + |y|^2)^{1+\alpha}}{1 + \lambda |y|^{2(1+\alpha)}} \right) + c, \quad (41)$$

$c \in \mathbb{R}, \lambda > 0$. A priori we do not know whether these critical points are minima for $J_{\bar{\rho}}$ (as it happens for $\alpha = 0$), so a direct application of 1.1 is not possible. However, we can still get the conclusion by comparing $J_{\bar{\rho}}(u_{\lambda,c})$ with the blow-up value provided by theorem 1.1.

Proof of theorem 1.4. Let us first compute $J(u_{\lambda,c})$. Let $\varphi_t : S^2 \rightarrow S^2$ be the conformal transformation defined by $\pi(\varphi_t(\pi^{-1}(y))) = ty$. It is not difficult to prove that $\forall t > 0$

$$J_{\bar{\rho}}(u) = J_{\bar{\rho}}(u \circ \varphi_t + (1 + \alpha) \log |\det d\varphi_t|);$$

in particular, since

$$u_{\lambda,c} = u_{1,0} \circ \varphi_{\lambda^{\frac{1}{2(1+\alpha)}}} + (1 + \alpha) \log |\det \varphi_{\lambda^{\frac{1}{2(1+\alpha)}}}| + c - \log \lambda,$$

we have that $J(u_{\lambda,c})$ does not depend on λ and c . Thus we may assume $\lambda = 1$ and $c = 0$. A simple computation shows that

$$\int_{S^2} h e^{u_{1,0}} dv_{g_0} = 4e^{2\alpha} \int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{(1 + |y|^{2(1+\alpha)})^2} dy = \frac{4e^{2\alpha}\pi}{1 + \alpha}. \quad (42)$$

Since $u_{1,0}(p_1) = 0$ and $u_{1,0}$ solves

$$-\Delta u_{1,0} = \omega h e^{u_{1,0}} - 2(1 + \alpha) \quad \text{with} \quad \omega := 2(1 + \alpha)^2 e^{-2\alpha}$$

one has

$$\int_{S^2} u_{1,0} dv_{g_0} = 4\pi \int_{S^2} \Delta u_{1,0} G_{p_1} dv_{g_0} = -4\pi\omega \int_{S^2} h e^{\mu_{1,0}} G_{p_1} dv_{g_0}$$

and

$$\begin{aligned} \frac{1}{2} \int_{S^2} |\nabla u_{1,0}|^2 dv_{g_0} + 2(1+\alpha) \int_{S^2} u_{1,0} dv_{g_0} &= \frac{1}{2}\omega \int_{S^2} h e^{\mu_{1,0}} u_{1,0} dv_{g_0} + (1+\alpha) \int_{S^2} u_{1,0} dv_{g_0} = \\ &= \frac{\omega}{2} \int_{S^2} h e^{\mu_{1,0}} (u_{1,0} - \bar{\rho} G_{p_1}) dv_{g_0}. \end{aligned} \quad (43)$$

Since

$$G_{p_1}(\pi^{-1}(y)) := \frac{1}{4\pi} \log(1 + |y|^2) - \frac{1}{4\pi}$$

we get

$$\begin{aligned} \int_{S^2} h e^{\mu_{1,0}} (u_{1,0} - \bar{\rho} G_{p_1}) &= 2(1+\alpha) \int_{S^2} h e^{\mu_{1,0}} dv_{g_0} - 8e^{2\alpha} \int_{\mathbb{R}^2} \frac{|y|^{2\alpha} \log(1 + |y|^{2(1+\alpha)})}{(1 + |y|^{2(1+\alpha)})^2} dy = \\ &= 8\pi e^{2\alpha} - \frac{8\pi e^{2\alpha}}{1+\alpha} \int_0^{+\infty} \frac{\log(1+s)}{(1+s)^2} ds = \frac{8\pi\alpha e^{2\alpha}}{1+\alpha}. \end{aligned} \quad (44)$$

Using (42), (43) and (44) we obtain

$$J(u_{\lambda,c}) = J(u_{1,0}) = 8\pi(1+\alpha)(\log(1+\alpha) - \alpha) \quad \forall \lambda > 0, c \in \mathbb{R}.$$

To conclude the proof it is sufficient to observe that $u_{\lambda,c}$ have to be minimum points for $J_{\bar{\rho}}$ that is

$$\inf_{H^1(S^2)} J_{\bar{\rho}} = 8\pi(1+\alpha)(\log(1+\alpha) - \alpha).$$

Indeed if this were false then $J_{\bar{\rho}}$ would have no minimum points but, by theorem 1.1, we would get

$$\inf_{H^1(S^2)} J_{\bar{\rho}} = 8\pi(1+\alpha)(\log(1+\alpha) - \alpha) = J(u_{\lambda,c}).$$

This is clearly a contradiction. □

Remark 5.2. *There is no need to assume $p_1 = -p_2$.*

Indeed given two arbitrary points $p_1, p_2 \in S^2$ with $p_1 \neq p_2$ it is always possible to find a conformal diffeomorphism $\varphi : S^2 \rightarrow S^2$ such that $\varphi^{-1}(p_1) = -\varphi^{-1}(p_2)$. Moreover one has

$$\widetilde{J}_{\bar{\rho}}(u) = \widetilde{J}_{\bar{\rho}}(u \circ \varphi + (1+\alpha) \log |\det d\varphi|) + c_{\alpha, p_1, p_2}$$

$\forall u \in H^1(S^2)$, where \widetilde{J} is the Moser-Trudinger functional associated to

$$\widetilde{h} = e^{-4\pi\alpha G_{\varphi^{-1}(p_1)} - 4\pi\alpha G_{\varphi^{-1}(p_2)}}.$$

and c_{α, p_1, p_2} is an explicitly known constant depending only on α, p_1 and p_2 . In particular one can still compute $\min_{H^1(S^2)} J_{\bar{\rho}}$ and describe the minimum points of $J_{\bar{\rho}}$ in terms of φ and the family (41).

Acknowledgements

The author would like to express his gratitude to Professor Andrea Malchiodi for many valuable discussions and for his guidance during the preparation of this work.

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